

Linear complementarity on simplicial cones and the congruence orbit of matrices *

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Abstract

The congruence orbit of a matrix has a natural connection with the linear complementarity problem on simplicial cones formulated for the matrix. In terms of the two approaches – the congruence orbit and the family of all simplicial cones – we give equivalent classification of matrices from the point of view of the complementarity theory.

1. Introduction

We use in this introduction some standard terms and notations which will also be specified in the next section.

Let be $K \subset \mathbb{R}^m$ a cone, $K^* \in \mathbb{R}^m$ be its dual, $M : \mathbb{R}^m \rightarrow \mathbb{R}^m$ a linear mapping and $q \in \mathbb{R}^m$. The problem

$$\text{LCP}(K, q, M) : \text{ find } x \in K \text{ with } Mx + q \in K^* \text{ and } \langle x, Mx + q \rangle = 0$$

is called *the linear complementarity problem on the cone K* . In the case $K = \mathbb{R}_+^m$ it is denoted by $\text{LCP}(q, M)$ and called *the classical linear complementarity problem*.

As one of the most important problems in optimization theory, the classical linear complementarity problem has a broad literature (see [1] and the literature therein).

*2010 AMS Subject Classification: 90C33, 15A04; Key words and phrases: linear complementarity, \mathbf{Q} -matrix, \mathbf{P} -matrix, congruence orbit

Despite of the important progress last decades in this field, it still in the center of interest nowadays.

Besides the classical case, the linear complementarity on Lorentz cone and the cone of positive semi-definite matrices emerged as an important topic in the previous decade [2,3].

When $K \subset \mathbb{R}^m$ is a simplicial cone, the linear complementarity problem can be transformed by a linear mapping in the classical one. But general simplicial cones can differ substantially from each other in some aspects, one of them being the projection onto the cone, the mapping playing an essential role in the solution of optimization problems. If the linear mapping M is given, such an approach relates the problem to the *congruence orbit* of M , i.e. to the set of maps

$$\mathcal{O}(M) = \{L^\top M L : L \in \text{GL}(m, \mathbb{R})\}, \quad (1)$$

where $\text{GL}(m, \mathbb{R})$ denotes the *general linear group* of \mathbb{R}^m , i.e., the group of invertible linear maps of the vector space \mathbb{R}^m .

Among other results, in this note we will show that $\text{LCP}(K, q, M)$ is feasible for an arbitrary simplicial cone $K \subset \mathbb{R}^m$ and an arbitrary $q \in \mathbb{R}^m$ if and only if M is a *positive definite* mapping [1], i.e., if $\langle Mx, x \rangle > 0$, $\forall x \in \mathbb{R}^m$, $x \neq 0$, which is equivalent to saying that this property holds for each member of $\mathcal{O}(M)$. It turns out that this property is also equivalent with the much stronger **P** and **Q**-properties of all members of $\mathcal{O}(M)$ and equivalently, with the corresponding properties of M for each simplicial cone K .

It is possible that some of the problems considered in the present note already occurred in a different setting in the huge literature on linear complementarity. Even so, the approach of considering classical **P** and **Q**-properties of a matrix for the *whole family* of simplicial cones and the relation with the congruence orbit of the matrix seems a novel approach which justifies our following investigation.

2. Terminology and notations

Denote by \mathbb{R}^m the m -dimensional Euclidean space endowed with the scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$, and the Euclidean norm $\|\cdot\|$ and topology this scalar product defines. Denote $\langle x, y \rangle = 0$ by $x \perp y$.

Throughout this note we shall use some standard terms and results from convex geometry (see e.g. [4]).

Let K be a *convex cone* in \mathbb{R}^m , i. e., a nonempty set with (i) $K + K \subset K$ and (ii) $tK \subset K$, $\forall t \in \mathbb{R}_+ = [0, +\infty)$. The convex cone K is called *pointed*, if $K \cap (-K) = \{0\}$. The cone K is *generating* if $K - K = \mathbb{R}^m$. K is generating if and only if $\text{int } K \neq \emptyset$.

A closed, pointed generating convex cone is called *proper cone*.

The set

$$K = \text{cone}\{x^1, \dots, x^m\} := \{t_1 x^1 + \dots + t_m x^m : t_i \in \mathbb{R}_+, i = 1, \dots, m\} \quad (2)$$

with x^1, \dots, x^m linearly independent vectors in \mathbb{R}^m is called a *simplicial cone*. A simplicial cone is proper.

The *dual* of the convex cone K is the set

$$K^* := \{y \in \mathbb{R}^m : \langle x, y \rangle \geq 0, \forall x \in K\},$$

with $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^m .

The cone K is called *self-dual* if $K = K^*$. If K is self-dual, then it is proper.

In all that follows we will suppose that \mathbb{R}^m is endowed with a Cartesian system having an orthonormal basis e^1, \dots, e^m and the elements $x \in \mathbb{R}^m$ are represented by the column vectors $x = (x_1, \dots, x_m)^\top$, with x_i the coordinates of x with respect to this basis. (That is, \mathbb{R}^m will be the vector space of m -dimensional column vectors.)

The set

$$\mathbb{R}_+^m = \{x = (x_1, \dots, x_m)^\top \in \mathbb{R}^m : x_i \geq 0, i = 1, \dots, m\}$$

is called the *non-negative orthant* of the above introduced Cartesian reference system. In fact

$$\mathbb{R}_+^m = \text{cone}\{e^1, \dots, e^m\}.$$

A direct verification shows that \mathbb{R}_+^m is a self-dual cone.

If K is the simplicial cone defined by (2) and A is the non-singular matrix transforming the basis e^1, \dots, e^m to the linear independent vectors x^1, \dots, x^m , then obviously

$$K = A\mathbb{R}_+^m. \quad (3)$$

For simplicity from now on we will call a convex cone simply cone.

3. Changing the cone linearly

Lemma 1 *Let $W \subset \mathbb{R}^m$ be a cone and $A \in \text{GL}(m, \mathbb{R})$. Then $K = AW$ is a cone too and $K^* = A^{-T}W^*$.*

Proof. The first assertion is trivial.

Take $y \in K^*$. This is equivalent to

$$\langle Aw, y \rangle = \langle w, A^\top y \rangle \geq 0, \forall w \in W.$$

Thus, $y \in K^* \iff A^\top y \in W^* \iff y \in A^{-T}W^*$. □

Corollary 1 *If $W^* = W$, $A \in \text{GL}(m, \mathbb{R})$ then*

$$K = AW \iff K^* = A^{-T}W.$$

If K is the simplicial cone (2), then, because of the representation (3) and the self-duality of \mathbb{R}_+^m , we have

$$K^* = A^{-T}\mathbb{R}_+^m.$$

4. Linear transformation of a cone and the complementarity problem

For the mapping $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ the *complementarity problem* $\text{CP}(F, K)$ is to find $x \in \mathbb{R}^m$ such that

$$K \ni x \perp F(x) \in K^*.$$

The solution set of $\text{CP}(F, K)$ will be denoted by $\text{SOL}(F, K)$. We have

$$x \perp y \iff Ax \perp A^{-\top}y. \quad (4)$$

Hence, by using Lemma 1 and (4), we conclude the following result:

Proposition 1 *If W is a cone, $A \in \text{GL}(m, \mathbb{R})$ and $K = AW$, then*

$$\text{SOL}(F, K) = A(\text{SOL}(A^\top F A, W)).$$

Proof. Indeed,

$$\begin{aligned} Ax \in A(\text{SOL}(A^\top F A, W)) &\iff x \in \text{SOL}(A^\top F A, W) \iff W \ni x \perp A^\top F(Ax) \in W^* \\ &\iff K \ni Ax \perp F(Ax) \in K^* \iff Ax \in \text{SOL}(F, K). \end{aligned}$$

□

5. The case of linear complementarity

The complementarity problem $\text{CP}(f, K)$ with $F(x) = Mx + q$, where $M \in \mathbb{R}^{m \times m}$ and $q \in \mathbb{R}^m$, will be denoted by $\text{LCP}(K, M, q)$ and called *linear complementarity problem*. Thus, the linear complementarity problem $\text{LCP}(K, M, q)$ is to find $x \in \mathbb{R}^m$ such that

$$K \ni x \perp Mx + q \in K^*.$$

The solution set of $\text{LCP}(K, M, q)$ will be denoted by $\text{SOL}(K, M, q)$. In this case Proposition 1 becomes

Proposition 2 *If W is a cone, $A \in \text{GL}(m, \mathbb{R})$ and $K = AW$, then*

$$\text{SOL}(K, M, q) = A(\text{SOL}(W, A^\top M A, A^\top q)).$$

If $K = \mathbb{R}_+^m$, then $\text{CP}(F, K)$, $\text{SOL}(F, K)$, $\text{LCP}(K, M, q)$ and $\text{SOL}(K, M, q)$ will simply be denoted by $\text{CP}(F)$, $\text{SOL}(F)$, $\text{LCP}(M, q)$ and $\text{SOL}(M, q)$, respectively.

6. The congruence orbit of a matrix and the complementarity problem

If A and B are in $\mathbb{R}^{m \times m}$, then they are *congruent* and we write $A \sim B$, if there exists $L \in \text{GL}(m, \mathbb{R})$ such that

$$L^\top AL = B,$$

that is B is in the congruence orbit $\mathcal{O}(A)$ of A defined at (1). Obviously, \sim is an equivalence relation and in this case $\mathcal{O}(B) = \mathcal{O}(A)$.

In the case of simplicial cones Proposition 2 reduces to

Proposition 3 *If $K = L\mathbb{R}_+^m$ is a simplicial cone then*

$$\text{SOL}(K, A, q) = A(\text{SOL}(\mathbb{R}_+^m, L^\top AL, A^\top q)).$$

Hence the linear complementarity problem on a simplicial cone is equivalent to the complementarity problem on the non-negative orthant, that is, to the classical linear complementarity problem.

Remark 1 *Proposition 3 shows that for linear complementarity problems with matrix A on simplicial cones the congruence orbit $\mathcal{O}(A)$ of A appears in a natural way.*

Definition 1 *Let A be a linear transformation. Then, we say that*

1. *A has the K -**Q**-property if $\text{LCP}(K, A, q)$ has a solution for all q .*
2. *A has the K -**P**-property if $\text{LCP}(K, A, q)$ has a unique solution for all q .*
3. *The \mathbb{R}_+^m -**Q**-property (\mathbb{R}_+^m -**Q**-property) is called **Q**-property, (**P**-property) and the matrix of the linear transformation A with the **Q**-property (**P**-property) is called **Q**-matrix (**P**-matrix) (for simplicity we denote a linear transformation and its matrix by the same letter).*
4. *A has the general feasibility property with respect to K denoted K -**FS**-property if*

$$(AK + q) \cap K^* \neq \emptyset, \quad \forall q \in \mathbb{R}^m.$$

*If $K = \mathbb{R}_+^m$, then the K -**FS**-property is called **FS**-property and it is characterized by the relation*

$$(A\mathbb{R}_+^m + q) \cap \mathbb{R}_+^m \neq \emptyset, \quad \forall q \in \mathbb{R}^m.$$

*The matrix A with the **FS**-property is called **FS**-matrix.*

Remark 2 Obviously, the \mathbf{P} -property of a matrix A implies its \mathbf{Q} -property, and its \mathbf{Q} -property implies its \mathbf{FS} -property as well. A classical result going back to the paper [5] asserts that A possesses the \mathbf{P} -property if and only if all its principal minors are positive. Theorem 3.1.6 in [1] asserts that a positive definite matrix possesses the \mathbf{P} -property.

The \mathbf{FS} -property of a matrix can be considered the weakest one in the context of linear complementarity. It is easy to see that the \mathbf{FS} -property (K - \mathbf{FS} -property) of the matrix A is equivalent to $-A\mathbb{R}_+^m + \mathbb{R}_+^m = \mathbb{R}^m$ ($-AK + K^* = \mathbb{R}^m$).

With the notations in the above definition we have

Proposition 4 If $A \in \mathbb{R}^{m \times m}$ has the K - \mathbf{Q} -property (K - \mathbf{P} -property) then $M = L^\top AL \in \mathcal{O}(A)$ has the LK - \mathbf{Q} -property (LK - \mathbf{P} -property).

Example 1 The congruence orbit of the identity

We have $\mathcal{O}(I) = \{L^\top L : L \in \text{GL}(m, \mathbb{R})\}$.

Hence, each member of $\mathcal{O}(I)$ is a symmetric positive definite matrix.

The following lemma is based on Example 1 and shows that if the congruence orbit of a matrix contains a positive definite matrix, then all of its matrices are positive definite.

Lemma 2 If $\mathcal{O}(A)$ contains a symmetric positive definite matrix, then

$$\mathcal{O}(A) = \mathcal{O}(I).$$

Proof. We can suppose that A itself is a symmetric positive definite matrix.

If we denote by R the square root of A [6], then we can write

$$\begin{aligned} \mathcal{O}(A) &= \{L^\top AL : L \in \text{GL}(m, \mathbb{R})\} = \{(RL)^\top (RL) : L \in \text{GL}(m, \mathbb{R})\} = \\ &= \{M^\top M : M \in \text{GL}(m, \mathbb{R})\} = \mathcal{O}(I). \end{aligned}$$

□

Obviously, a symmetric positive definite matrix is nothing else but a symmetric \mathbf{P} -matrix. Hence, by Lemma 2 we conclude

Corollary 2 Each member of the congruence orbit of a symmetric positive definite matrix is a \mathbf{P} -matrix.

How about the congruence orbit of a non-symmetric \mathbf{P} -matrix? Can it have a property similar to the one stated in Corollary 2? We will show that this holds if and only if the matrix is positive definite.

Lemma 3 If the diagonal of the matrix $A \in \mathbb{R}^{m \times m}$ contains some non-positive element, then $\mathcal{O}(A)$ contains non- \mathbf{FS} -matrices.

Proof. (a) Let D_k be a diagonal matrix with $d_{kk} = -1$ and $d_{ii} = 1$ if $i \neq k$. If $A = (a_{ij})_{i,j=1,\dots,m}$, then $B = D_k^\top A D_k$ is a matrix with $b_{ij} = a_{ij}$ if $i \neq k$ and $j \neq k$, $b_{ki} = -a_{ki}$, $i \neq k$, $b_{jk} = -a_{jk}$, $j \neq k$, and $b_{kk} = a_{kk}$.

(b) Without loss of generality we can assume that $a_{11} \leq 0$. Assume that the positive terms of the first line of A are a_{1j} , a_{1k} , \dots , a_{1l} . Then if D is the diagonal matrix with $d_{ii} = -1$, $i \in \{j, k, \dots, l\}$ and $d_{ii} = 1$, $i \notin \{j, k, \dots, l\}$, then using the remark at (a) we conclude that

$$C = D^\top A D$$

is a matrix whose first line contains only non-positive elements. Hence, for any $q = (-1, *, \dots, *)^\top$ we have $(C\mathbb{R}_+^m + q) \cap \mathbb{R}_+^m = \emptyset$. \square

As we have stated at Remark 2 a positive definite matrix possesses the **P**-property, but simple examples show, that the converse is not true.

We have the obvious assertion:

Lemma 4 *If $A \in \mathbb{R}^{m \times m}$ is not positive definite, then its symmetrizer*

$$S(A) = \frac{A + A^\top}{2} \tag{5}$$

is not positive definite neither.

Remark 3 *Observe that the diagonal of $S(A)$ defined as in (5) coincides with the diagonal of A .*

Theorem 1 *Let $A \in \mathbb{R}^{m \times m}$. Then the following assertions are equivalent:*

1. *Each member of $\mathcal{O}(A)$ is a **FS**-matrix.*
2. *Each member of $\mathcal{O}(A)$ is a **Q**-matrix.*
3. *Each member of $\mathcal{O}(A)$ is a **P**-matrix.*
4. *A is a positive definite matrix.*

*Hence, if any of the above conditions hold then A possesses the K -**P**, K -**Q**, K -**FS**-properties for any simplicial cone K .*

Proof. Suppose that assertion 1 hold.

(a) Assume that A is not positive definite, that is, item 4. does not hold. Then, by Lemma 4, the same is true for $S(A)$. That is, $S(A)$ is a symmetric matrix which is not positive definite. Then, it has non-positive eigenvalues, that is, in the spectral decomposition

$$S(A) = O D O^\top$$

D is a diagonal matrix with some non-positive elements. On the other hand we have

$$D = O^\top S(A)O = \frac{O^\top AO + O^\top A^\top O}{2}.$$

Now, since $O^\top A^\top O = (O^\top AO)^\top$, it follows, according to Remark 3, that the diagonal of D coincides with the diagonal of $O^\top AO$. Since this diagonal contains non-positive elements, it follows by Lemma 3 that the orbit $\mathcal{O}(A)$ contains elements which are not **FS**-matrices. This shows the implication $1 \Rightarrow 4$.

(b) If 4 holds, then for each $L \in \text{GL}(m, \mathbb{R})$ and any $x \neq 0$ we have

$$\langle L^\top ALx, x \rangle = \langle ALx, Lx \rangle > 0,$$

that is, $L^\top AL$ is positive definite and hence, by Theorem 3.1.6 in [1] is a **P**-matrix, and hence also a **Q**-matrix and an **FS**-matrix. Thus, we have the implications $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

(c) The last assertion of the theorem follows from Propositions 3 and 4. \square

Lemma 5 *If for $A \in \text{GL}(m, \mathbb{R})$ there exists $x \in \mathbb{R}^m$ with $\langle Ax, x \rangle > 0$, then $\mathcal{O}(A)$ contains a matrix with at least one positive element in its diagonal.*

Proof. Let $S(A)$ be the symmetrizer of A . Then $\langle S(A)x, x \rangle > 0$.

Consider the spectral decomposition

$$S(A) = ODO^\top$$

of $S(A)$. Then,

$$\langle S(A)x, x \rangle = \langle ODO^\top x, x \rangle = \langle DO^\top x, O^\top x \rangle > 0$$

Hence, the diagonal matrix D must contain positive elements, since $\langle Dy, y \rangle = \sum_i d_{ii}y_i^2 > 0$ with $y = O^\top x$.

Now,

$$D = \frac{O^{-1}A(O^\top)^{-1} + O^{-1}A^\top(O^\top)^{-1}}{2} = \frac{O^{-1}A(O^\top)^{-1} + (O^{-1}A(O^\top)^{-1})^\top}{2},$$

hence the diagonal of $O^{-1}A(O^\top)^{-1}$ coincides with the diagonal of D and hence it must contain positive elements. \square

The matrix $A = (a_{ij})_{i,j=1,\dots,m} \in \mathbb{R}^{m \times m}$ is called *positive*, if $a_{ij} > 0$, $\forall i, j$.

Lemma 6 *For the matrix $A \in \mathbb{R}^{m \times m}$ the following two assertions are equivalent:*

1. $\mathcal{O}(A)$ contains a matrix with a positive element on its diagonal,

2. $\mathcal{O}(A)$ contains a positive matrix.

Proof. (a) We first prove that if the matrix $A := (a_{ij})_{i,j=1,\dots,m} \in \mathbb{R}^{m \times m}$ contains a positive principal submatrix of order $n - 1 < m$, then it has a conjugate containing a positive principal submatrix of order n .

Suppose that $A(1 : n, 1 : n) := (a_{ij})_{i,j=1,\dots,n}$ has the property that $a_{ij} > 0$ whenever $i, j \in \{2, \dots, n\}$ and let $A(2 : n, 2 : n) = (a_{ij})_{i,j=2,\dots,n}$.

Denote by $I \in \mathbb{R}^{n \times n}$ the unit matrix and let $E_{12} \in \mathbb{R}^{n \times n}$ be the matrix with 1 in the position $(i, j) = (1, 2)$ and 0 elsewhere. Let $L_t = I + tE_{12}$ with t a real parameter. Put

$$B = L_t A(1 : n, 1 : n) L_t^\top = (b_{ij})_{i,j=1,\dots,n}$$

and $B(2 : n, 2 : n) = (b_{ij})_{i,j=2,\dots,n}$. Then, we have

$$B(2 : n, 2 : n) = A(2 : n, 2 : n), \quad (6)$$

$$b_{11} = a_{11} + ta_{21} + ta_{12} + t^2 a_{22}, \quad (7)$$

$$b_{1i} = a_{1i} + ta_{2i}, \quad i \geq 2, \quad (8)$$

$$b_{i1} = a_{i1} + ta_{i2}, \quad i \geq 2. \quad (9)$$

From (6), (7), (8) and (9), it follows that for $t > 0$ large enough we will have $b_{ij} > 0$, $i, j = 1, \dots, n$.

(b) Applying the procedure from (a), the element $a_{ii} > 0$ on the diagonal of A can be augmented to obtain in $\mathcal{O}(A)$ a matrix with positive principal minor of order 2, then a matrix with positive principal minor of order 3 in $\mathcal{O}(A)$, and so on, to obtain a positive matrix in $\mathcal{O}(A)$. \square

Theorem 2 *If for $A \in \text{GL}(m, \mathbb{R})$ there exists an $x \in \mathbb{R}^m$ with $\langle Ax, x \rangle > 0$ and an $y \in \mathbb{R}^m \setminus \{0\}$ with $\langle Ay, y \rangle \leq 0$, then $\mathcal{O}(A)$ contains non-**FS**-matrices and **Q**-matrices as well.*

Proof. By Theorem 1 $\mathcal{O}(A)$ must contain non-**FS**-matrices.

By Lemma 5, $\mathcal{O}(A)$ must contain a matrix B with at least one positive element on its diagonal. Then, by Lemma 6, $\mathcal{O}(B) = \mathcal{O}(A)$ must contain a positive matrix. By Theorem 3.8.5 in [1], it follows that a such matrix is a **Q**-matrix. \square

Corollary 3 *Suppose that $A \in \text{GL}(m, \mathbb{R})$. Then, exactly one of the following alternatives hold:*

1. $\langle Ax, x \rangle > 0, \forall x \in \mathbb{R}^m \setminus \{0\} \iff$ each member of $\mathcal{O}(A)$ is a **FS**-matrix \iff each member of $\mathcal{O}(A)$ is a **Q**-matrix \iff each member of $\mathcal{O}(A)$ is a **P**-matrix.

2. If $\langle Ax, x \rangle \leq 0, \forall x \in \mathbb{R}^m$, then no matrix in $\mathcal{O}(A)$ can have the **FS**-property.
3. If for some non-zero elements x and y in \mathbb{R}^m one has $\langle Ax, x \rangle > 0$ and $\langle Ay, y \rangle \leq 0$, then $\mathcal{O}(A)$ contains non-**FS**-matrices and **Q**-matrices as well.

Proof. Only item 2 needs proof. Let $A \in \text{GL}(m, \mathbb{R})$ a matrix with $\langle Ax, x \rangle \leq 0, \forall x \in \mathbb{R}^m$. Suppose to the contrary that there is a matrix $L^\top AL \in \mathcal{O}(A)$ which has the **FS**-property. Let $q \in \mathbb{R}^m$ be a vector with all components negative. Then, there exists $x \in \mathbb{R}_+^m$ such that

$$L^\top ALx + q \in \mathbb{R}_+^m. \quad (10)$$

Thus,

$$0 \leq \langle x, L^\top ALx + q \rangle = \langle Lx, ALx \rangle + \langle x, q \rangle. \quad (11)$$

If $x \neq 0$, then the right hand side of equation (11) is negative, which is a contradiction. Hence, $x = 0$. Then, (10) implies $q \in \mathbb{R}_+^m$, which contradicts the choice of q . In conclusion, no matrix in $\mathcal{O}(A)$ can have the **FS**-property. \square

The alternatives listed in Corollary 3 can be formulated in complementarity terms too:

Corollary 4 *Suppose that $A \in \text{GL}(m, \mathbb{R})$. Then exactly one of the following alternatives hold :*

1. A possesses the K -**FS**- property for any simplicial cone $K \iff A$ possesses the K -**Q**- property for any simplicial cone $K \iff A$ possesses the K -**P**- property for any simplicial cone K .
2. There is no simplicial cone K for which A possesses the K -**FS**-property.
3. There exists a simplicial cone K for which A does not have the K -**FS**-property and there exists a simplicial cone L for which A possesses the L -**Q**-property.

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